Mechanical Waves

Ripples on a lake, musical sounds, seismic tremors triggered by an earthquake—
all these are *wave* phenomena.

Waves can occur whenever a system is disturbed from equilibrium and when the disturbance can travel, or *propagate,* from one region of the system to another.

Wave = *propagation of an oscillation*

As a wave propagates, it carries energy. The energy in light waves from the sun warms the surface of our planet; the energy in seismic waves can crack our planet's crust.

Types of waves:

Mechanical waves—waves that travel within some material called a *medium.*

Electromagnetic waves—including light, radio waves, infrared and ultraviolet radiation, and x rays—can propagate even in empty space, where there is *no* medium

LEARNING GOALS

- What is meant by a mechanical wave, and the different varieties of mechanical waves.
- How to use the relationship among speed, frequency, and wavelength for a periodic wave.
- How to interpret and use the mathematical expression for a sinusoidal periodic wave.
- How to calculate the speed of waves on a rope or string.
- How to calculate the rate at which a mechanical wave transports energy.
- What happens when mechanical waves overlap and interfere.
- The properties of standing waves on a string, and how to analyze these waves.
- How stringed instruments produce sounds of specific frequencies.

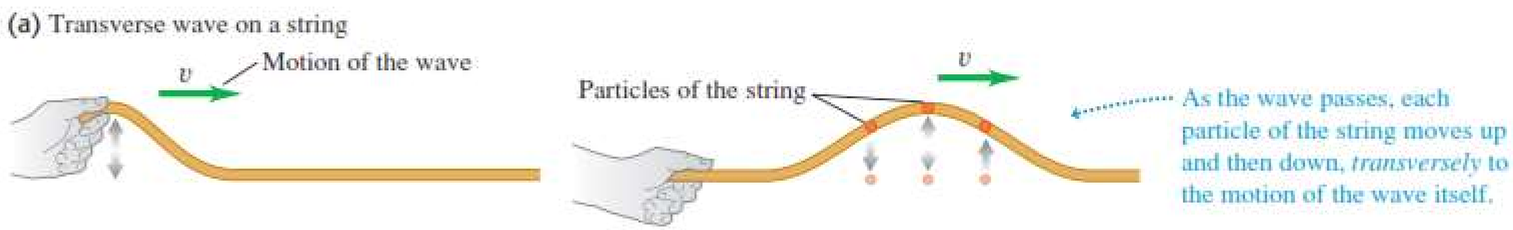
I. Types of Mechanical Waves

A **mechanical wave** = disturbance that travels through some material or substance called **medium.**

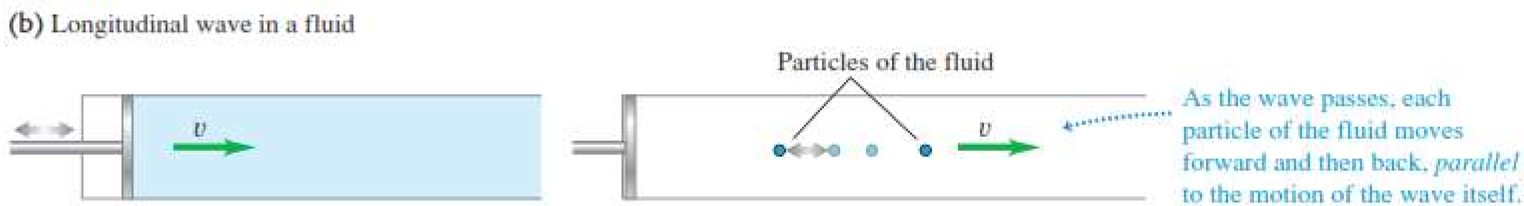
As the wave travels through the medium, the particles that make up the medium undergo displacements of various kinds, depending on the nature of the wave.

Three varieties of mechanical waves:

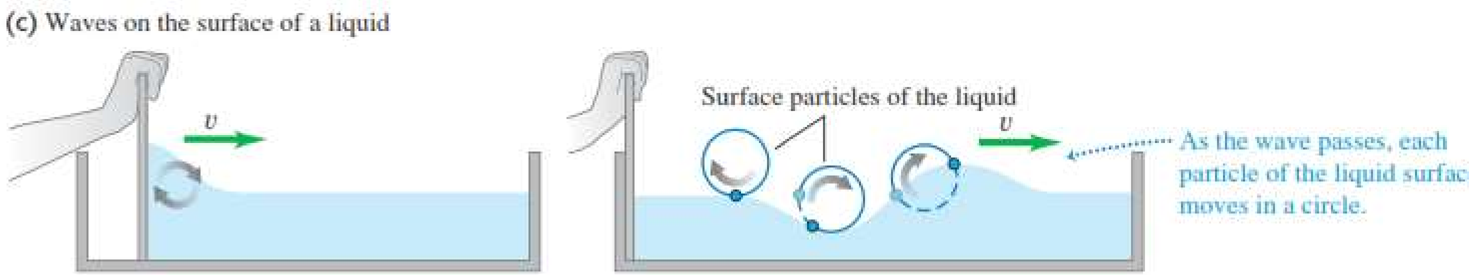
1. Transverse wave: the displacements of the medium are perpendicular or *transverse to the direction of travel of the wave along the medium.*



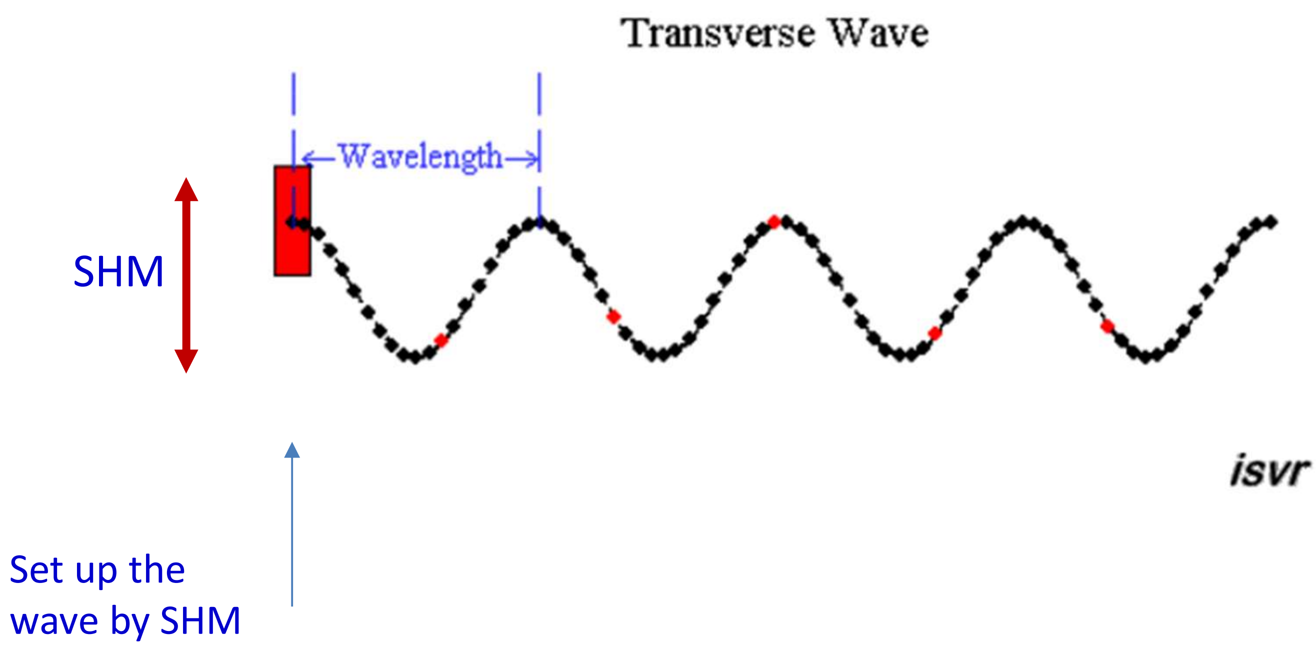
2. Longitudinal wave: the motions of the particles of the medium are back and forth along the *same direction that the wave travels.*



3. wave with *both longitudinal and transverse components.*



Transverse wave

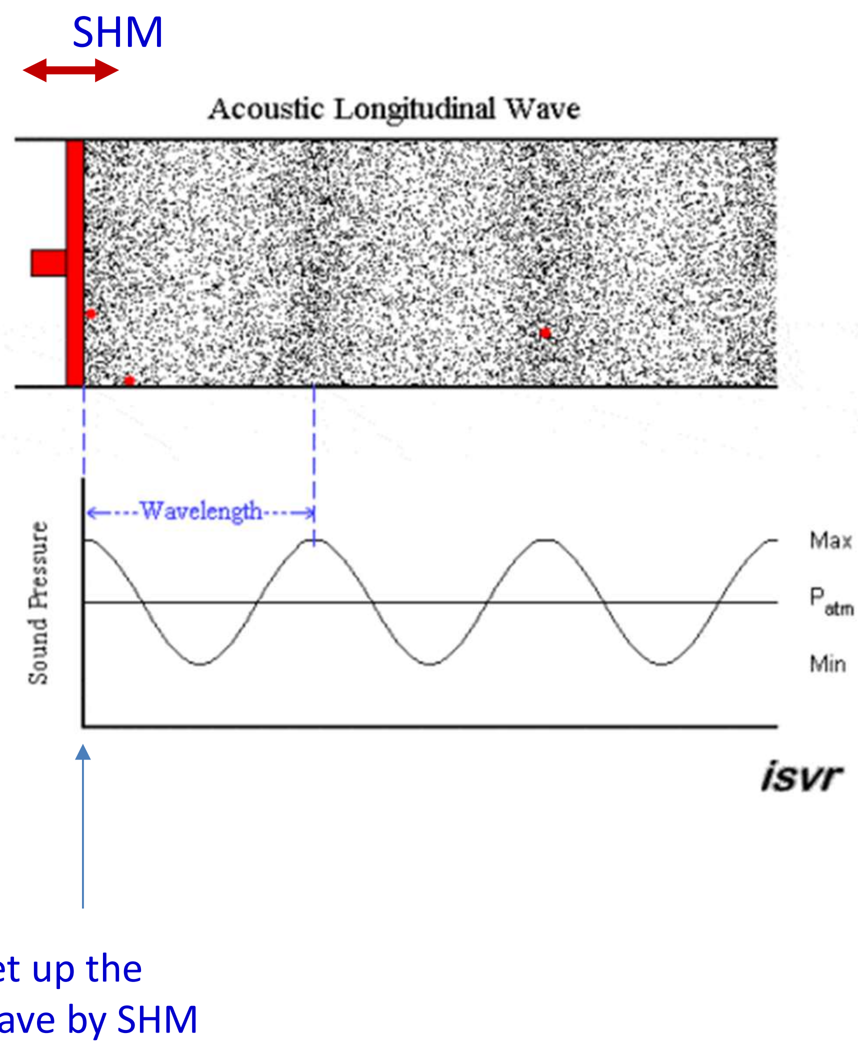


Oscillation

That propagates with *v* in the medium

⇔ particles from successive regions start to oscillate

Longitudinal wave



Oscillation

That propagates with *v* in the medium

⇔ particles from successive regions start to oscillate

Observe:

- oscillation of medium particles for L and T waves
- once the wave passed the particles return to equilibrium

The different types pf waves have some common characteristics:

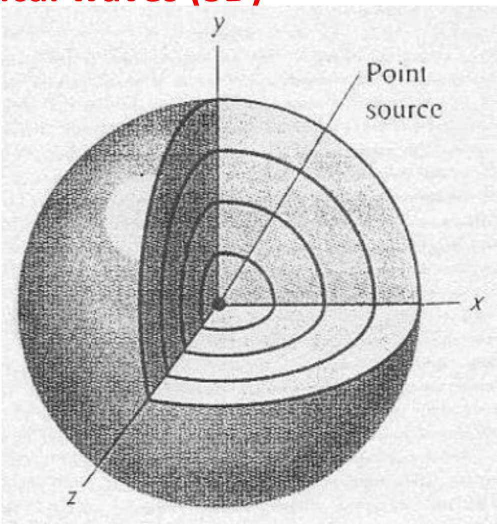
Wave front: the continuous line or surface including all the points in space reached by the wave at the same instant through the medium

As a function of the shape of the wave front the waves can be classified in:

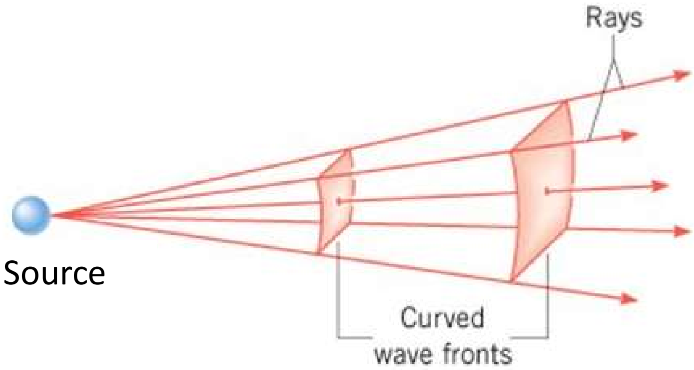
Circular waves (2D)

Perturbation initiated in the center

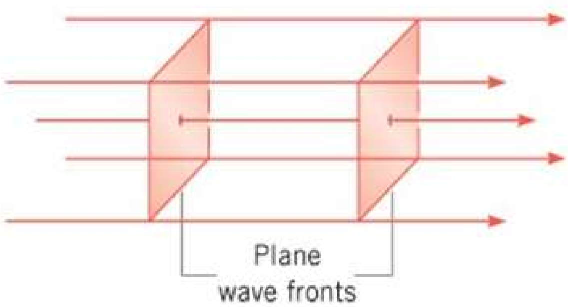
Spherical waves (3D)



Plane waves



At large distance from the source, the wave fronts become less and less curved => Flat wave front surfaces
=> plane waves



Waves characteristics:

1. Wave speed [*v*]

in each case the disturbance travels or *propagates* with a definite speed through the medium. This speed is called the speed of propagation, or simply the **wave speed. Its value is determined** in each case by the mechanical properties of the medium.

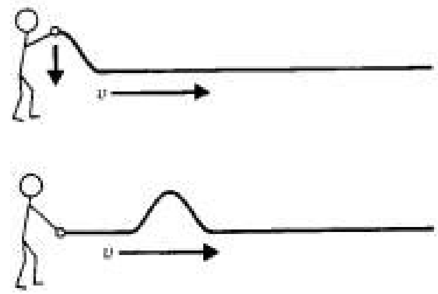
The wave speed is not the same as the speed with which particles move when they are disturbed by the wave (see later).

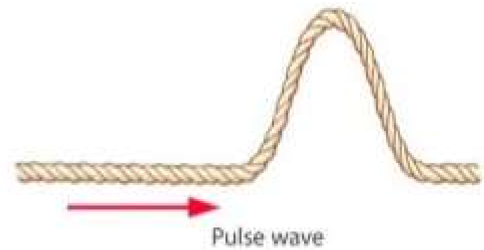
2. The medium itself does not travel through space; its individual particles undergo back-and-forth or up-and-down motions around their equilibrium positions. The overall pattern of the wave disturbance is what travels.

3. **Waves transport energy, but not matter, from one region to another**. To set any of these systems into motion, we have to put in energy by doing mechanical work on the system. The wave motion transports this energy from one region of the medium to another.

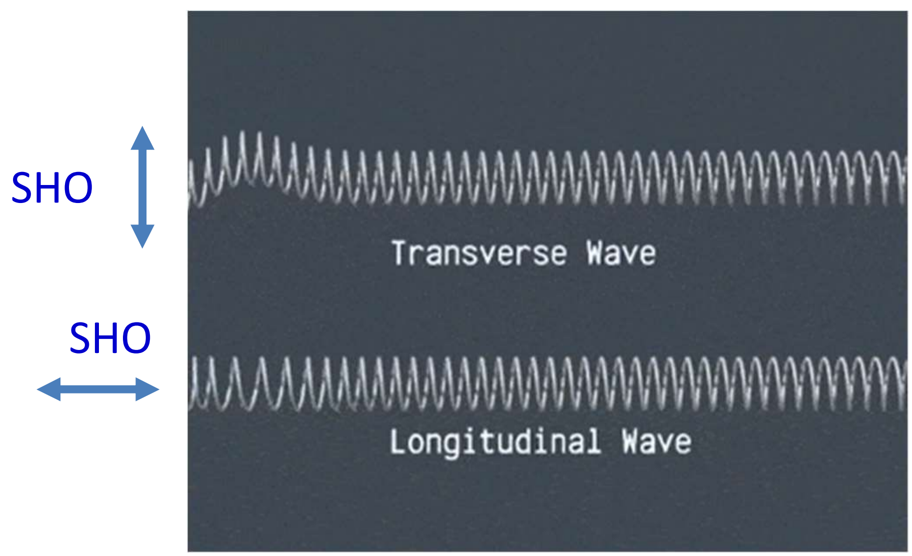
II. Periodic Waves

The transverse wave on a stretched string in is an example of **a wave pulse**. The hand shakes the string up and down just once, exerting a transverse force on it as it does so. The result is a single “wiggle,” or pulse, that travels along the length of the string. The tension in the string restores its straight-line shape once the pulse has passed. => **source =non-periodic disturbance**

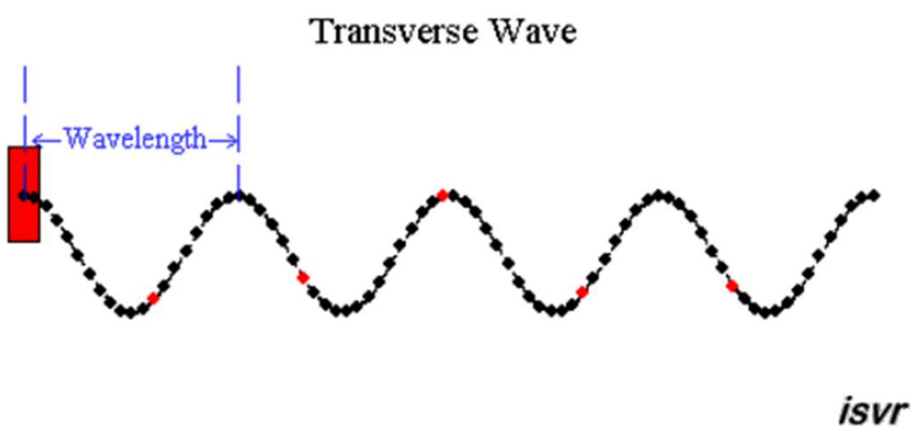


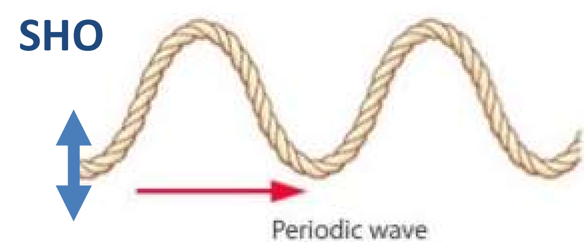


Longitudinal vs transverse wave pulse in a string



A more interesting situation develops when we give the free end of the string a repetitive, or *periodic, motion.* Then each particle in the string also undergoes periodic motion as the wave propagates, and we have a **periodic wave. => source = periodic oscillation (SHO)**



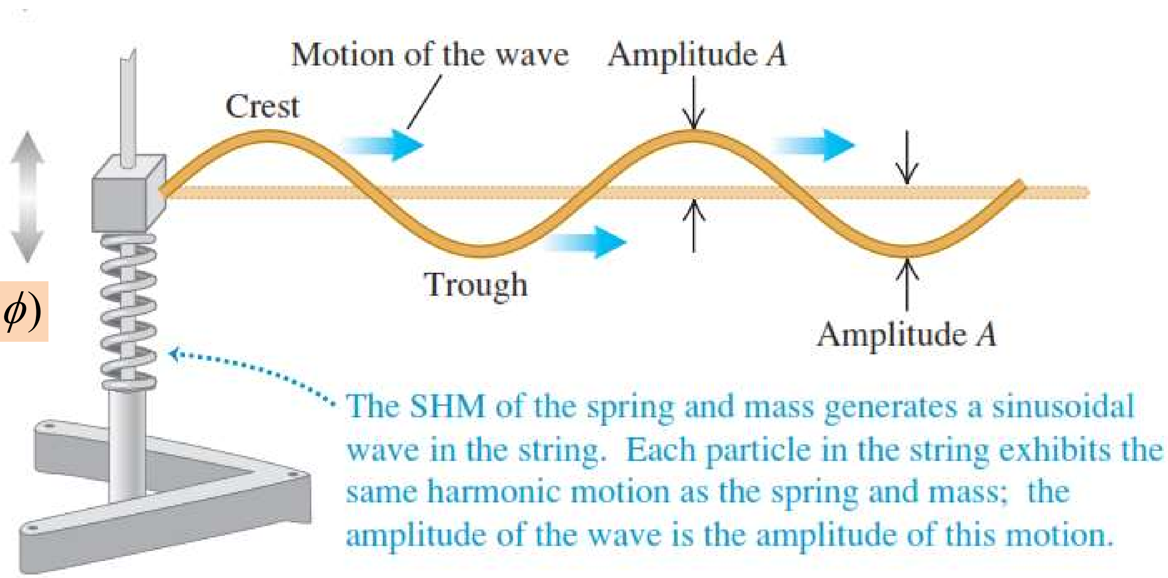


Periodic Transverse Waves

we move the end of the string up and down with *simple harmonic motion (SHM) with amplitude A, frequency f angular frequency* $\omega=2\pi f$ *and* period $T=1/f$. The wave that results is a symmetrical sequence of *crests and troughs.*

SHO

$$y(t) = A\cos(\omega t + \phi)$$



A block of mass *m attached to a spring undergoes simple harmonic motion, producing* a sinusoidal wave that travels to the right on the string. (In a real-life system a driving force would have to be applied to the block to replace the energy carried away by the wave.)

Periodic waves with SHM → **sinusoidal waves**

Obs. Any periodic wave can be represented as a combination of sinusoidal waves.

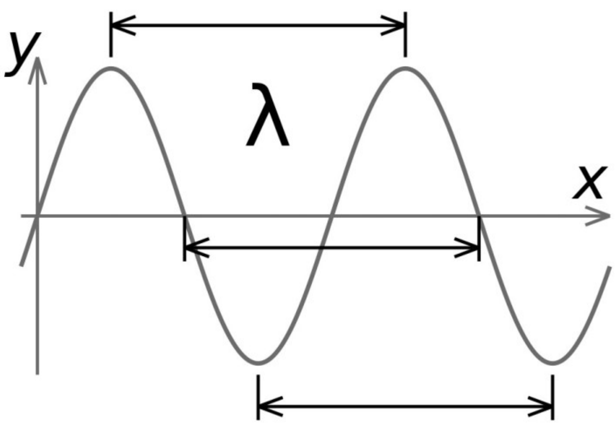
When a sinusoidal wave passes through a medium, every particle in the medium undergoes simple harmonic motion with the same frequency.

Wave motion vs. particle motion

! Distinguish between the motion of the transverse wave along the string and the motion of a particle of the string. The wave moves with constant speed v along the length of the string, while the motion of the particle is simple harmonic and transverse (perpendicular) to the length of the string.

Wavelength

For a periodic wave, the shape of the string at any instant is a repeating pattern
=> **wavelength of the wave,** denoted by λ **[m]**



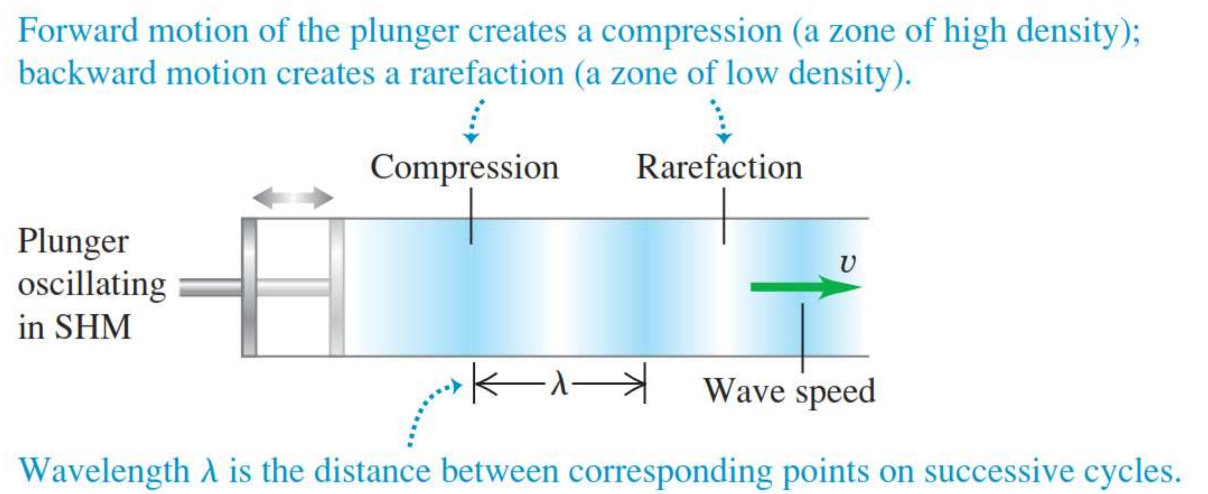
The wave pattern travels with constant speed v and advances a distance of one wavelength in a time interval of one period *T.*

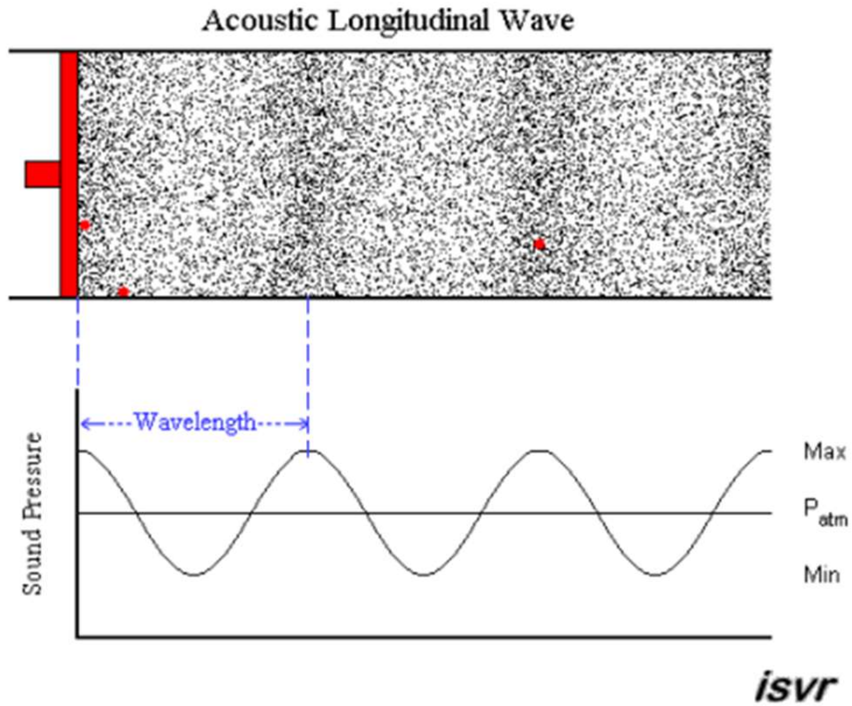
$$v = \frac{\lambda}{T} = \lambda f$$

Obs. Waves on a string propagate 1D but all the concepts remain valid for 2D, 3D cases

Periodic Longitudinal Waves

Using an oscillating piston to make a sinusoidal longitudinal wave in a fluid.





Sound wave = longitudinal wave in air (fluid) –*see next courses… Acoustics*

III. Mathematical Description of a Wave

Many characteristics of periodic waves can be described by using the concepts of wave speed, amplitude, period, frequency, and wavelength. Often, though, we need a more detailed description of the positions and motions of individual particles of the medium at particular times during wave propagation.

wave function $\Psi=\Psi(x,t)$

For a transverse wave y=y(x,t) describes the displacement y of points along x axis at time t

From this we can find the velocity and acceleration of any particle, the shape of the string, and anything else we want to know about the behavior of the string at any time.

Wave Function for a Sinusoidal Wave

General characteristics for any type of periodical wave

$$y(x) = y(x+\lambda)$$

$$y(x,t) = y(x,t+T)$$

Wave = periodical phenomenon in space and time

$$y(x,t) = A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \quad \text{(sinusoidal wave moving in } +x\text{-direction)}$$

It's convenient to define a quantity k, called the **wave number:**

$$k = \frac{2\pi}{\lambda} \qquad \text{(wave number)}$$

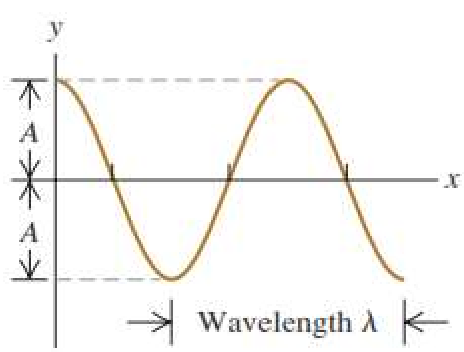
Substituting $\lambda = 2\pi/k$ and $f = \omega/2\pi$ into the wavelength–frequency relationship $v = \lambda f$ gives

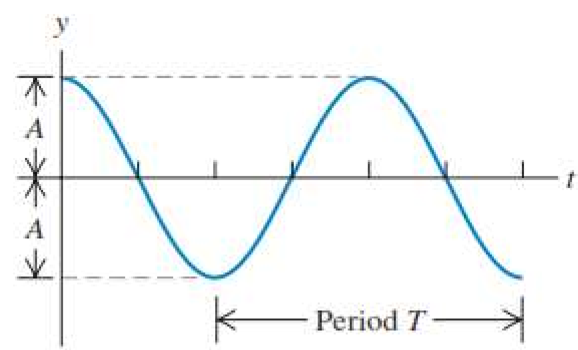
$$\omega = vk \qquad \text{(periodic wave)}$$

$$y(x, t) = A\cos(kx - \omega t) \qquad \text{(sinusoidal wave moving in } +x\text{-direction)}$$

$$y(x, t) = A\cos(kx + \omega t) \qquad \text{(sinusoidal wave moving in - x direction)}$$

Graphing the Wave Function





$$y(x, t = 0) = A\cos kx = A\cos 2\pi\frac{x}{\lambda}$$

Shape of the string at *t =0.*

$$y(x = 0, t) = A\cos(-\omega t) = A\cos\omega t = A\cos 2\pi\frac{t}{T}$$

Displacement of the particle at *x =0 as a function of time*

$$y(x,t) = A\cos(kx \pm \omega t)$$

The quantity **(kx±ωt)** is called **the phase**
Plays the role of an angular quantity [rad]

Its value for any values of *x and t determines what part of the sinusoidal cycle is* occurring at a particular point and time.

Crest: y=A; cos **(kx±ωt)** =1 => phase =0,2π, 4π…(2n π)
Trough: y=-A, cos **(kx±ωt)** =-1 => phase =π, 3π, 5π…(2n+1) π

The wave speed

Is the speed with which we have to move along with the wave to keep alongside a point of a given phase, such as a particular crest of a wave on a string.

For a wave traveling in the that means **(kx-ωt) = ** constant.

Taking the derivative with respect to t: $k\dfrac{dx}{dt} = \omega$ ⇔ $\dfrac{dx}{dt} = \dfrac{\omega}{k} = v$ Speed of wave or **phase speed**

Equivalent definitions $v = \dfrac{\omega}{k} = \dfrac{\lambda}{T} = \lambda f$

Differential equation of the wave

Particle Velocity and Acceleration in a Sinusoidal Wave

Particle Velocity

From *y(x,t)* one can deduce the transverse velocity of a particle in a transverse wave $v_y(x,t)$

$$y(x,t) = A\cos(kx - \omega t) \quad \Rightarrow \quad v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

-periodical function => SHM
-maximum value ($v_y^{max} = \pm\omega A$)
-may be larger, equal or smaller than the wave speed v, depending on A and ω

Particle Acceleration

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t)$$

Equivalent to what we got for SHM

We can also compute partial derivatives of y(x,t) with respect to *x, holding t* constant.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$$

From:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t)$$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$$

And: $\omega = vk$ $\Rightarrow$ $$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

$\Leftrightarrow$ $$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad \text{(wave equation)}$$

- **one of the most important** equations in all of physics
- valid in the most general situation, whether the wave is periodical or not
- electric and magnetic field satisfy wave equation with v=c (speed of light) –light is an electromagnetic wave

Generalizing propagation in a 3D medium

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{\partial^2\Psi(x,t)}{\partial y^2} + \frac{\partial^2\Psi(x,t)}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\Psi(t)}{\partial t^2} = 0$$

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ $\Rightarrow$ $$\Delta\Psi(x,t) - \frac{1}{v^2}\frac{\partial\Psi(x,t)}{\partial t} = 0$$

Laplace 2nd order differential operator

IV. Speed of a wave

One of the key properties of any wave is the wave *speed.*
Light waves in air have a much greater speed of propagation than do sound waves in air 3.10^8m/s vs 344m/s; that's why you see the flash from a bolt of lightning before you hear the clap of thunder.

We would like to correlate the speed of a wave in a medium and some characteristic properties of the medium

it turns out that for many types of mechanical waves, including waves on a string, the expression for wave speed has the same general form:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

Transverse waves in a string

The restoring force is the tension in the string F, it tends to bring back the string in the unperturbed position
The inertia resisting to the return to equilibrium is the mass of the string, more precisely the mass/unit length

$\mu = dm / dx$ or $\mu = m / l$ ⇒ $$v = \sqrt{\frac{F}{\mu}}$$

Longitudinal sound waves

The gas pressure provides the force that tends to return the gas to its undisturbed state when a sound wave passes through. The inertia is provided by the density, or mass per unit volume, of the gas.

⇒ $$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma p_0}{\rho}}$$

B = the bulk modulus of the medium
P_0-equillibrium pressure of the gas
γ =ratio of heat capacities

See next chapter Acoustics for details

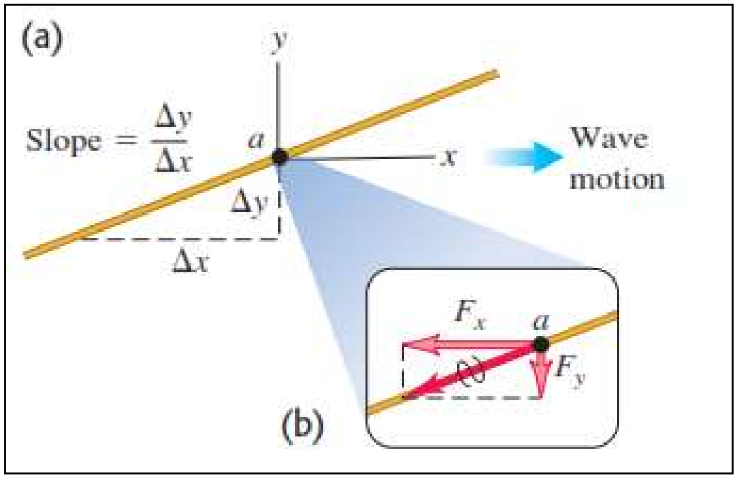
V. Energy in Wave Motion

Propagating waves carry energy in their propagation.

To produce a wave we apply a force to a portion of the wave medium; the point where the force is applied moves, so we do *work on the system. As the wave propagates, each portion* of the medium exerts a force and does work on the adjoining portion

Transverse waves on a string:

a) Point *a on a string carrying a* wave from left to right.
(b) The components of the force exerted on the part of the string to the right of point *a* by the part of the string to the left of point *a.*



$$\frac{F_y}{F} = -slope = -\frac{\partial y(x,t)}{\partial x}$$

When point a moves in the *y*-direction, the force *Fy* does work on this point

⇒ therefore transfers energy into the part of the string to the right of a.

⇒ the corresponding power *P* (rate of doing work) at the point a is
the transverse force *Fy(x,t)* at a times the transverse velocity $v_y(x,t)=\partial y(x,t)/\partial t$ of that point :

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

For a sinusoidal wave: $y(x,t) = A\cos(kx - \omega t)$

➡ $\dfrac{\partial y(x,t)}{\partial x} = -kA\sin(kx - \omega t)$ $\dfrac{\partial y(x,t)}{\partial t} = \omega A\sin(kx - \omega t)$

➡ $$P(x,t) = Fk\omega A^2\sin^2(kx - \omega t)$$

By using the relationships $\omega = vk$ and $v^2 = F/\mu$, we can also express

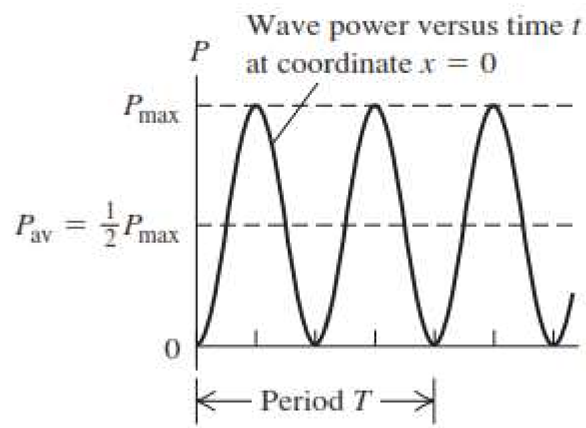
$$P(x,t) = \sqrt{\mu F}\,\omega^2 A^2\sin^2(kx - \omega t)$$

The **maximum value** of the instantaneous power occurs when the $\sin^2$ function =1

$$P_{max} = \sqrt{\mu F}\,\omega^2 A^2$$

To obtain the **average power**, we note that:
<$\sin^2$ function >$_{\text{over any whole number of cycles}}$ =1/2

➡ $$P_{av} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$$



The average rate of energy transfer is: ~$\omega^2 A^2$ valid for any mechanical wave

for electromagnetic waves: ~ **A^2 independent on ω**

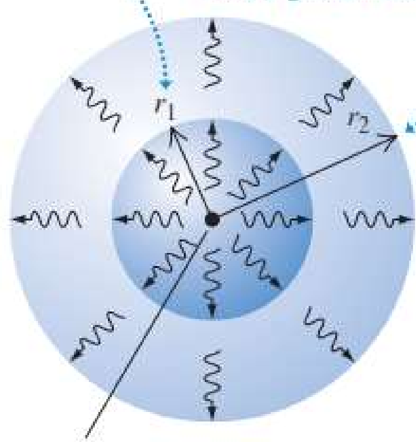
Wave Intensity

Waves on a string carry energy in just one dimension of space (along the direction of the string). But other types of waves, including sound waves in air and seismic waves in the body of the earth, carry energy across all three dimensions of space.

For waves that travel in three dimensions, we define the **intensity** (denoted by *I) to be the time average rate at which energy is transported by the wave, per unit area, across a surface perpendicular to the direction of propagation.*

$$I = \frac{\text{average power}}{\text{unit area}} \left[\frac{W}{m^2}\right]$$

If waves spread out equally in all directions from a source, the intensity at a distance *r from the source is inversely proportional to* r^2



The greater the distance from a wave source, the greater the area over which the wave power is distributed and the smaller the wave intensity.

If the power output of the source is *P, then the* average intensity through a sphere with radius r and surface area $4\pi r^2$ is: $I = P / 4\pi r^2$.

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \qquad \text{(inverse-square law for intensity)}$$

See later (Acoustics) the geometrical attenuation of 3D waves

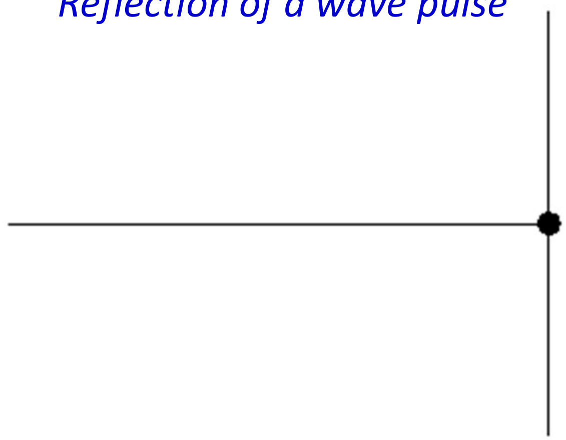
VI. Wave Interference, Boundary Conditions, and Superposition

The **interference** represents the *overlapping of waves* that happen when several waves pass through the same area at the same time.

Up to this point we discussed waves that propagate continuously in the same direction.

But when a wave strikes the boundaries of its medium, all or part of the wave is ***reflected.***

Reflection of a wave pulse



The conditions at the end of the string, such as a rigid support or the complete absence of transverse force, are called **boundary conditions.**

The initial and reflected waves overlap in the same region of the medium. This overlapping of waves is called **interference.**

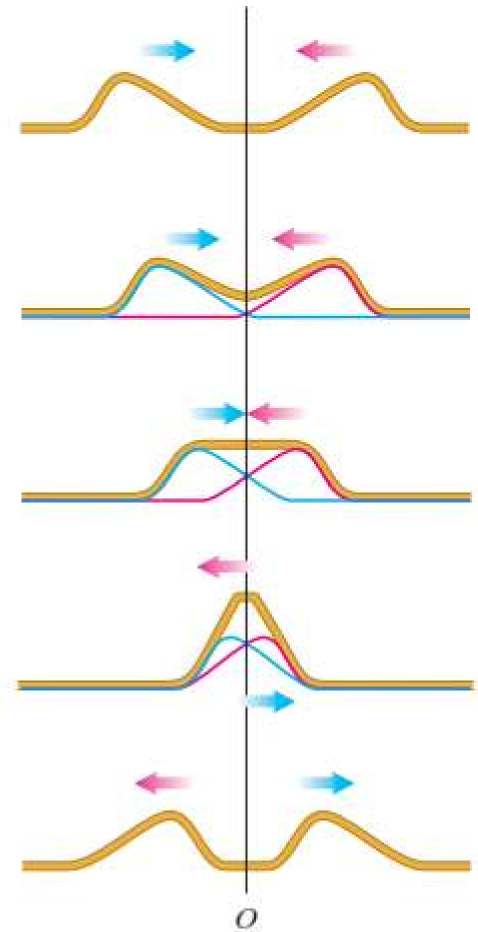
The Principle of Superposition

Combining the displacements of the separate pulses at each point to obtain the actual displacement

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$y(x, t) = y_1(x, t) + y_2(x, t) \qquad \text{(principle of superposition)}$$

Overlap of two wave pulses—both right side up—traveling in opposite directions.



Resulting oscillation is the **vector sum** of individual oscillations

The principle of superposition is of central importance in all types of waves. Superposition also applies to electromagnetic waves (such as light) and many other types of waves.

When a friend talks to you while you are listening to music, you can distinguish the sound of speech and the sound of music from each other. This is precisely because *the total sound wave reaching your ears is the algebraic sum of the wave* produced by your friend's voice and the wave produced by the speakers of your stereo.

In some special conditions, two sound waves did *not* combine in this simple linear way => the sound you would hear in this situation would be a hopeless jumble.

INTERFERENCE OF OSCILLATIONS

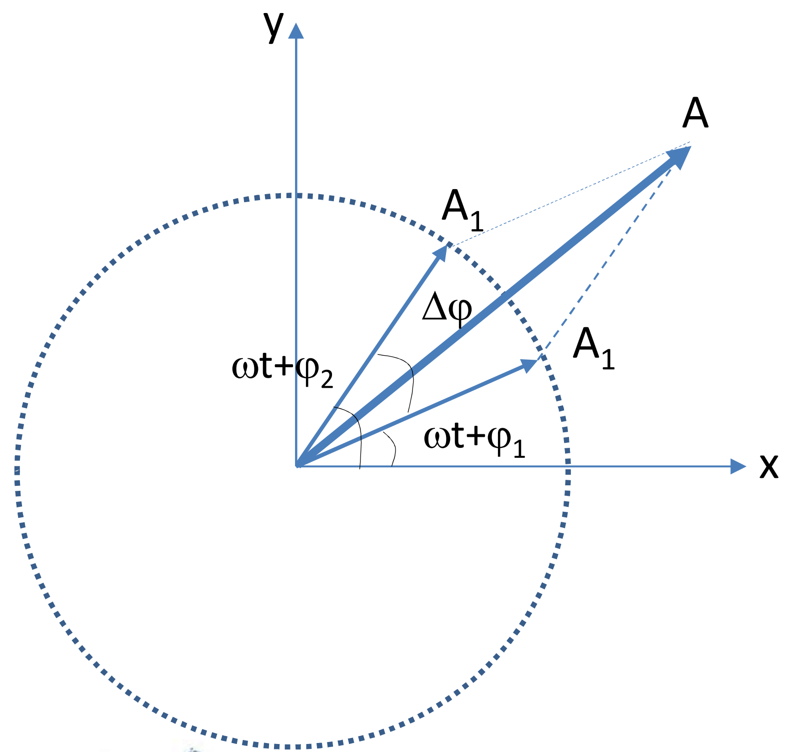
Analysis using **phasor representation**

$$\begin{cases} y_1(t) = A\sin(\omega_1 t + \varphi_1); \\ y_2(t) = A\sin(\omega_2 t + \varphi_2) \end{cases}$$

$$y(t) = y_1(t) + y_2(t)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\Delta\varphi)$$

$$\Delta\varphi = (\omega_2 - \omega_1)t + (\varphi_2 - \varphi_1)$$



Value of average $\langle A^2 \rangle \sim$ wave intensity $\dot{I}$

$$\langle A^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2 \frac{1}{T}\int_0^T \cos\left[(\omega_2-\omega_1)t + (\varphi_2 - \varphi_1)\right]dt$$

the integral is $= 0$ for $\omega_1 \neq \omega_2$

in this case, no interference and

$A^2 = A_1^2 + A_2^2 \Leftrightarrow$ $\underline{\dot{I} = \dot{I}_1 + \dot{I}_2}$

$$\langle A^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2 \frac{1}{T}\int_0^T \cos\left[(\omega_2-\omega_1)t + (\varphi_2-\varphi_1)\right]dt$$

If $\omega_1 = \omega_2$, in order to have interference one must have: $\varphi_1 - \varphi_2 =$ constant in time

$\Rightarrow$ coherent waves (phase difference const. in time)

Then $\langle A^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2 \cos(\varphi_2 - \varphi_1)$

- $\Delta\varphi = \varphi_2 - \varphi_1 = 2n\pi$ $\Rightarrow \cos(\varphi_2-\varphi_1) = 1$

  $n = 0, 1, 2, \ldots$ maximum of interference

  $I \sim A^2 = max = (A_1 + A_2)^2$

- $\Delta\varphi = \varphi_2 - \varphi_1 = (2n+1)\pi$ $\Rightarrow \cos(\varphi_2-\varphi_1) = -1$

  $n = 0, 1, \ldots$

  $I \sim A^2 = min = (A_1 - A_2)^2$

In terms of λ ; $\varphi = \omega t - Kx$

$\Delta\varphi = 2n\pi \Rightarrow \Delta\varphi = K\Delta x = 2n\pi$

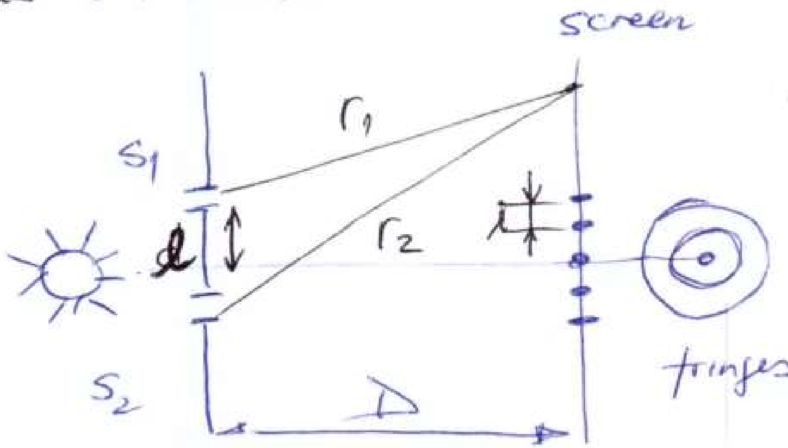
$= \frac{2\pi}{\lambda}\Delta x = 2n\pi$

$\Rightarrow \boxed{\Delta x = 2n\frac{\lambda}{2}}$ maximum of interference

analogously $\boxed{\Delta x = (2n+1)\frac{\lambda}{2}}$ minimum of interference

Obs Interference is a phenomenon common to waves, independent on their nature

Experiment of Thomas Young



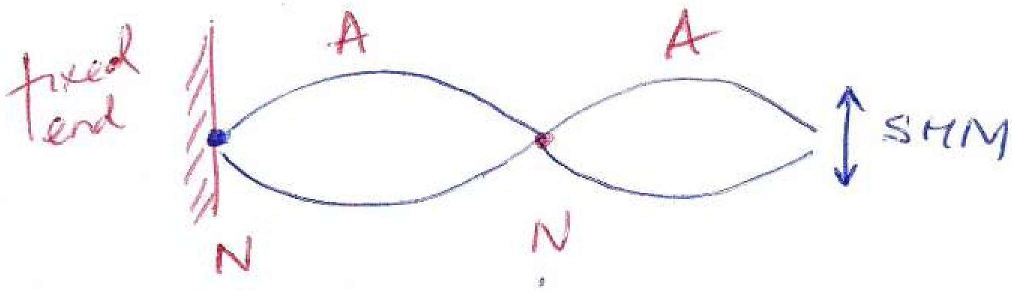
max. and min fringes on screen

$i = d_{n+1} - d_n$

$= \frac{\lambda D}{\ell}$

VII. Standing Waves on a String

String fixed at its left end. Its right end is moved up and down in SHM to produce a wave that travels to the left; the wave reflected from the fixed end travels to the right.



N = **nodes:** points at which the string never moves

A = **antinodes:** points at which the amplitude of string motion is greatest

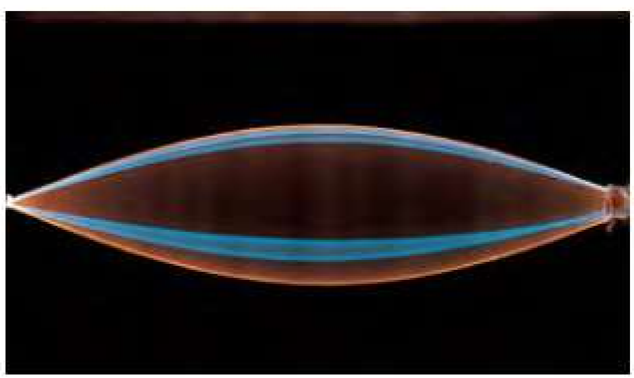
consider the superposition of two waves propagating through the string:

one representing the original or incident wave and

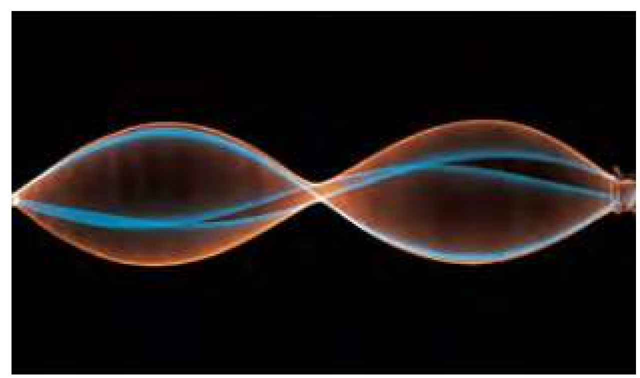
the other representing the wave reflected at the fixed end.

Standing waves

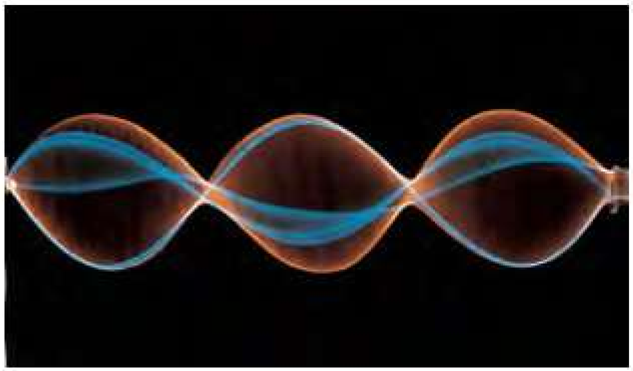
(a) String is one-half wavelength long.



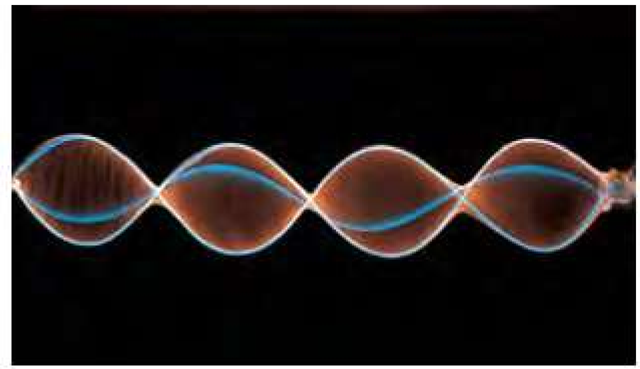
(b) String is one wavelength long.



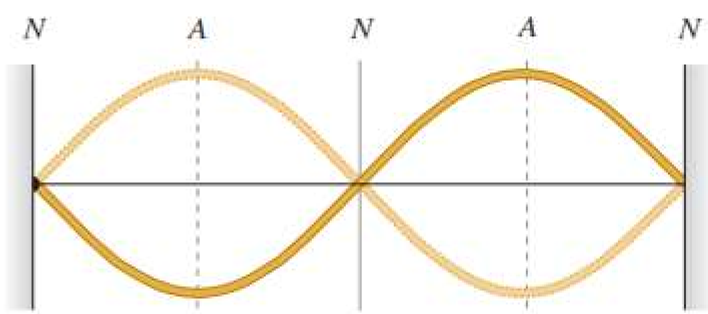
(c) String is one and a half wavelengths long.



(d) String is two wavelengths long.



(e) The shape of the string in (b) at two different instants

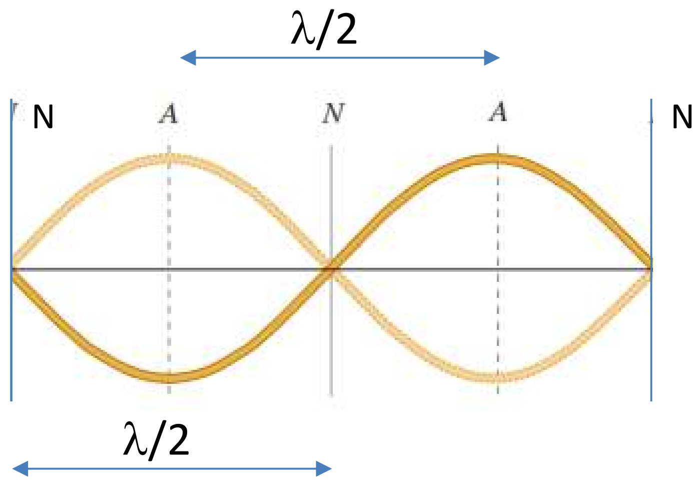


N = **nodes:** points at which the string never moves

A = **antinodes:** points at which the amplitude of string motion is greatest

Because the wave pattern doesn't appear to be moving in either direction along the string, it is called a **standing wave.**

(To emphasize the difference**, a wave that *does move*** along the string is called a **traveling wave.)**



The distance between successive nodes or between successive antinodes is one half-wavelength $\lambda/2$

From the **principle of superposition:**

N: At a node the displacements of the two waves (incident + reflected) are always equal and opposite and cancel each other out. This cancellation is called **destructive interference.**

A: At the antinodes the displacements of the two waves (incident + reflected) are always identical, giving a large resultant displacement; this phenomenon is called **constructive interference.**

We can derive a wave function for the standing wave by adding the wave functions $y_1(x,t)$ and $y_2(x,t)$ for two waves with equal amplitude, period, and wavelength traveling in opposite directions.

the wave reflected from a fixed end of a string is inverted, so we give a negative sign to one of the waves:

$$y_1(x, t) = -A\cos(kx + \omega t) \quad \text{(incident wave traveling to the left)}$$
$$y_2(x, t) = A\cos(kx - \omega t) \quad \text{(reflected wave traveling to the right)}$$

$$\Rightarrow \quad y(x, t) = y_1(x, t) + y_2(x, t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$$

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$y(x, t) = y_1(x, t) + y_2(x, t) = (2A\sin kx)\sin\omega t$$

$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \quad \text{(standing wave on a string, fixed end at } x = 0\text{)}$$

$$A_{SW} = 2A$$

The standing-wave amplitude A_{SW} is twice the amplitude *A of either of the original* traveling waves

$$y(x,t) = (A_{SW} \sin kx) \sin \omega t \quad \text{(standing wave on a string, fixed end at } x = 0\text{)}$$

$$y(x,t) = [\text{Function of x}] \times [\text{Function of time}]$$

A_{SW} sin(kx)
at each instant of time the shape of the string is the same

But unlike a travelling wave the standing wave stays in the same position oscillating up and down as described by the *sin(ωt)* factor

Each point in the string still undergoes simple harmonic motion, but all the points between any successive pair of nodes oscillate *in phase.*

The **position of nodes**: $\sin kx = 0$ $\quad kx = 0, \pi, 2\pi, 3\pi, \ldots,$

or, using $k = 2\pi/\lambda$,

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots$$
$$= 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$$

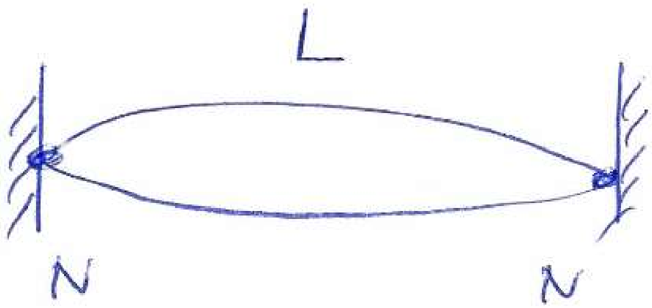
(nodes of a standing wave on a string, fixed end at $x = 0$)

A standing wave, unlike a traveling wave, does not transfer energy from one end to the other. The two waves that form it would individually carry equal amounts of power in opposite directions. There is a local flow of energy from each node to the adjacent antinodes and back, but the average rate of energy transfer is zero at every point => the average power is zero

VIII. Normal Modes of a String

Let's now consider a string of a definite length *L,* rigidly held at *both ends.*

Such strings are found in many musical instruments, including pianos, violins, and guitars. When a guitar string is plucked, a wave is produced in the string; this wave is reflected and re-reflected from the ends of the string, making a standing wave. This standing wave on the string in turn produces a sound wave in the air, with a frequency determined by the properties of the string. This is what makes stringed instruments so useful in making music.



Both ends fixed =>
Boundary conditions: Nodes at each end

The condition for nodes: $L = n\frac{\lambda}{2} \quad (n = 1, 2, 3, \ldots)$ (string fixed at both ends)

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots)$$

Waves can exist on the string if the wavelength is not equal to one of these values, but there cannot be a steady wave pattern with nodes and antinodes, and the total wave cannot be a standing wave.

Corresponding to the series of possible standing-wave wavelengths λ_n

$$\Rightarrow \quad f_n = v/\lambda_n.$$

The smallest frequency f_1 corresponds to the wavelength (n=1); λ_1=2L

$$f_1 = \frac{v}{2L}$$

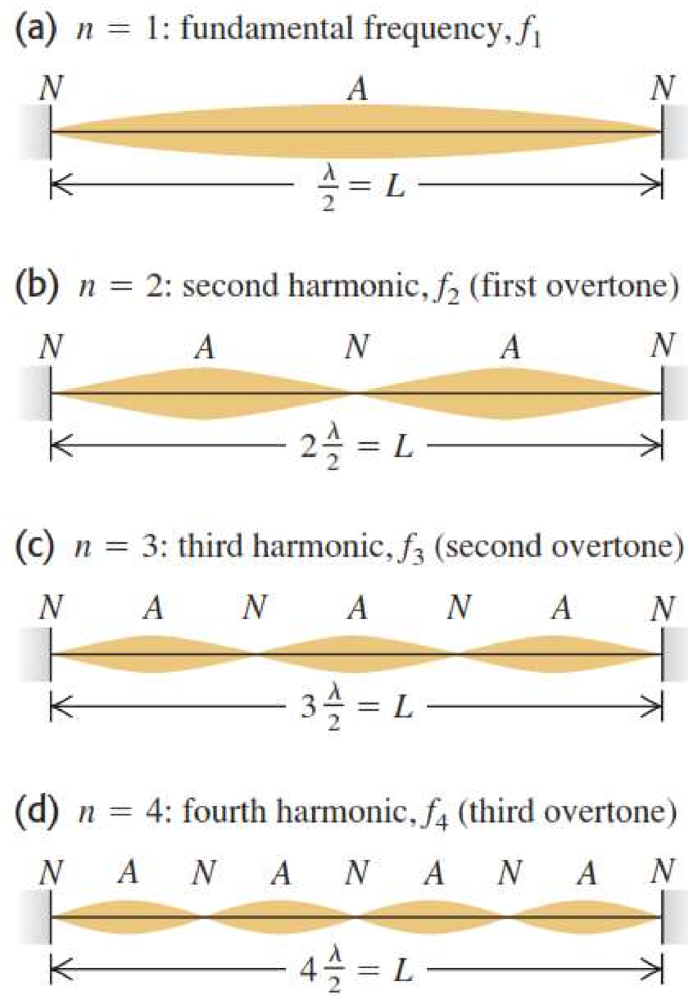
called the fundamental frequency

The other standing-wave frequencies $f_n = nv/2L$ are called **harmonics**

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots)$$

Musicians sometimes call them **overtones**

The first harmonic is the same as the fundamental frequency



The first four normal modes of a string fixed at both ends.

For a string with fixed at x=0 and x=L ends the wave function *y(x,t)* of the *nth standing wave is given by:*

$$y_n(x, t) = A_{SW} \sin k_n x \sin \omega_n t$$

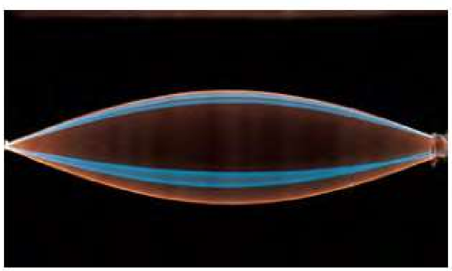
Which satisfies the condition that there is a node at x=0

$$\omega = \omega_n = 2\pi f_n \text{ and } k = k_n = 2\pi/\lambda_n$$

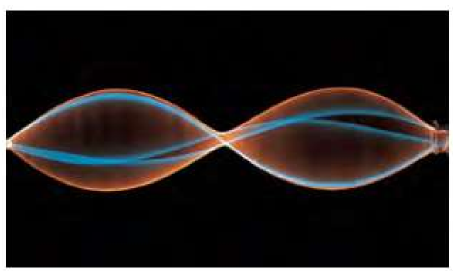
A **normal mode of an oscillating system is a motion in which all particles** of the system move sinusoidally with the same frequency.

There are infinitely many normal modes, each with its characteristic frequency and vibration pattern.

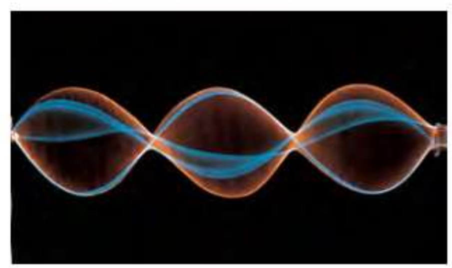
Stroboscopic images



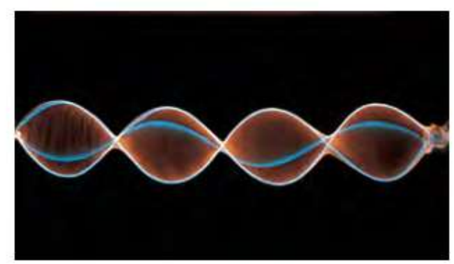
Fundamental =first harmonic



2nd harmonic



3rd harmonic



4th harmonic

Standing Waves and String Instruments

As we have seen, the fundamental frequency of a vibrating string is $f_1 = v/2L$

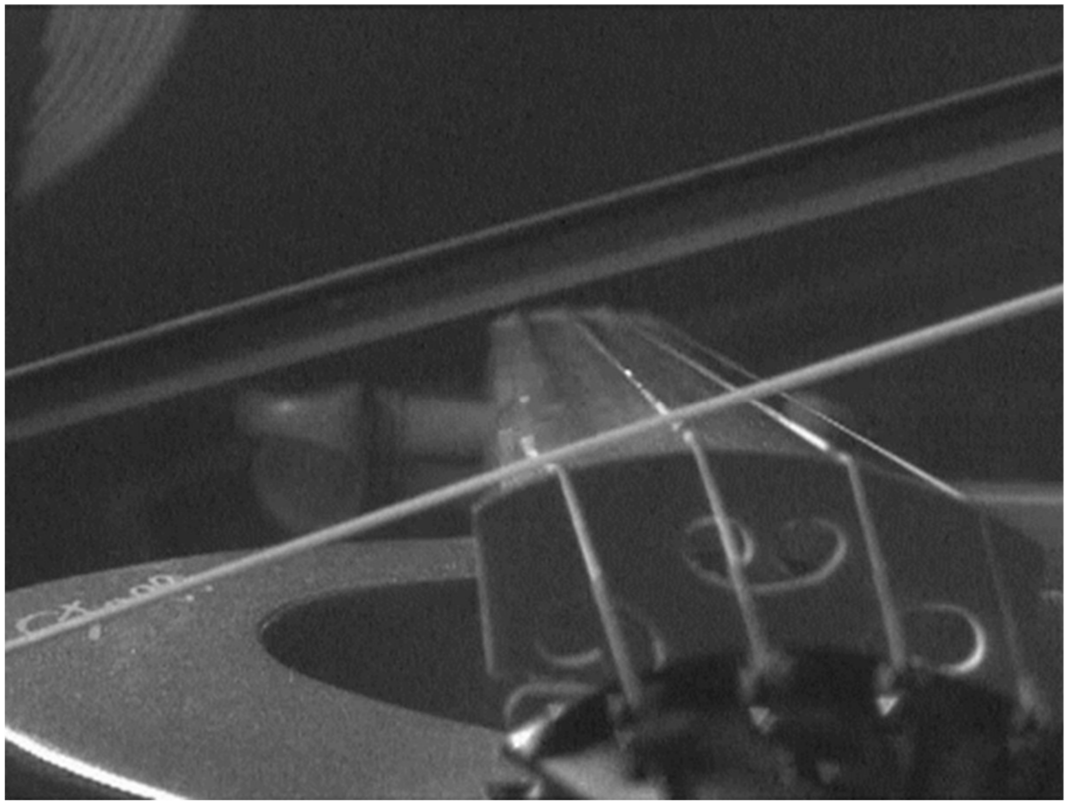
The speed of waves on the string is determined by: $v = \sqrt{F/\mu}$

$$\Rightarrow \quad f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

the fundamental frequency of the sound wave created in the surrounding air by the vibrating string

depends on the properties of the string

The inverse dependence of frequency on length *L is illustrated by the long strings of the bass (low-frequency) section of* the piano or the bass viol compared with the shorter strings of the treble section of the piano or the violin. The pitch of a violin or guitar is usually varied by pressing a string against the fingerboard with the fingers to change the length *L of the vibrating portion of the string.* Increasing the tension *F increases the* wave speed and thus increases the frequency (and the pitch). All string instruments are “tuned” to the correct frequencies by varying the tension; you tighten the string to raise the pitch. Finally, increasing the mass per unit length decreases the wave speed and thus the frequency. The lower notes on a steel guitar are produced by thicker strings, and one reason for winding the bass strings of a piano with wire is to obtain the desired low frequency from a relatively short string.

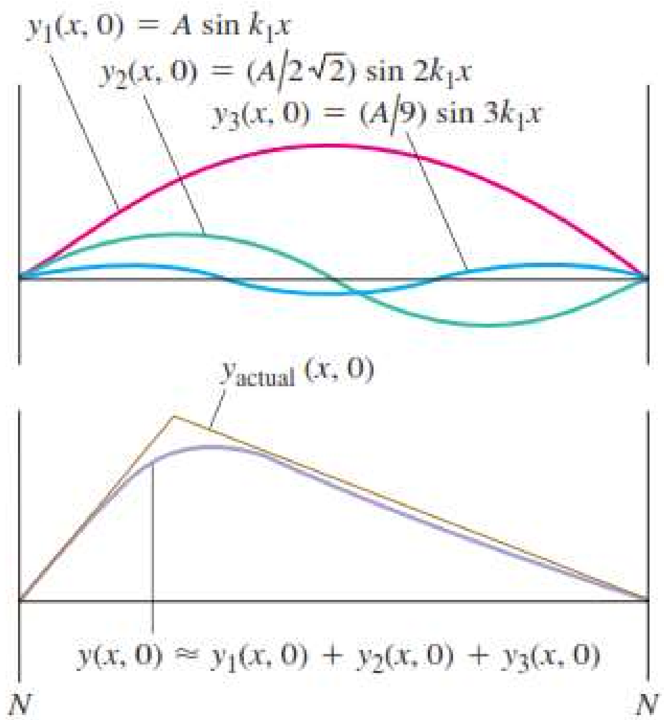


Complex Standing Waves

If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode. Such a vibrating string would displace the surrounding air with the same frequency, producing a traveling sinusoidal sound wave that your ears would perceive as a **pure tone**.

But when a string is struck (as in a piano) or plucked (as is done to guitar strings), the shape of the displaced string is not as simple => The fundamental as well as many overtones are present in the resulting vibration. This motion is therefore a combination or *superposition of many normal* modes.

Several simple harmonic motions of different frequencies are present simultaneously, and the displacement of any point on the string is the sum (or superposition) of the displacements associated with the individual modes.

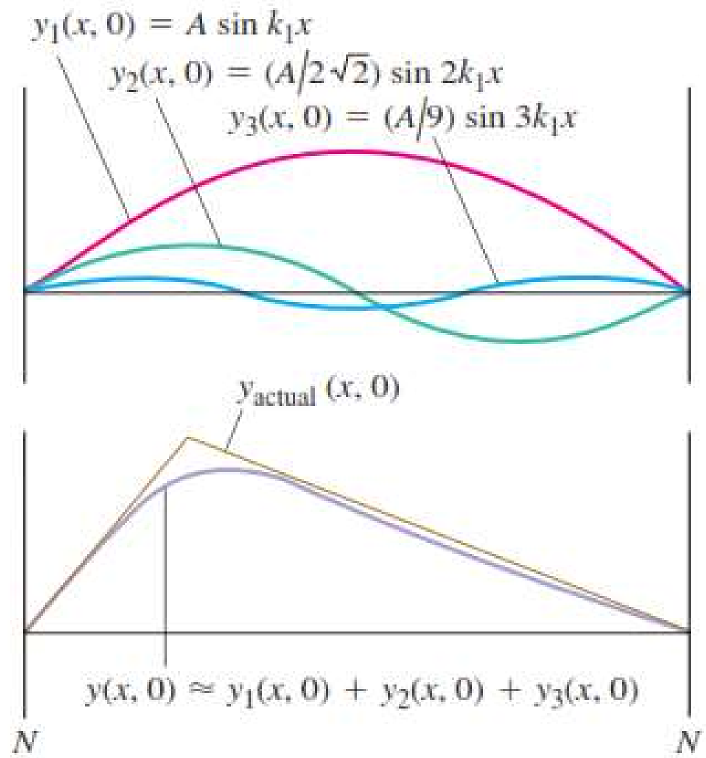


When a guitar string is plucked (pulled into a triangular shape) and released, a standing wave results. The standing wave is well represented (except at the sharp maximum point) by the sum of just three sinusoidal functions. Including additional sinusoidal functions further improves the representation.

The sound produced by the vibrating string is likewise a superposition of traveling sinusoidal sound waves, which you perceive as a rich, complex tone with the fundamental frequency.

The standing wave on the string and the traveling sound wave in the air have similar **harmonic content** (the extent to which frequencies higher than the fundamental are present).
The harmonic content depends on how the string is initially set into motion.

It is possible to represent every possible motion of the string as some superposition of normal-mode motions. Finding this representation for a given vibration pattern is called *harmonic analysis.*



The sum of sinusoidal functions that represents a complex wave is called a *Fourier series*
=> Fourier series decomposition

$$y(x,t) = \sum_{j=1}^{n} \left[A_j \cos(jkx) + B_j \sin(jkx) \right]$$